ECONOMIC LOT SCHEDULING

Economic lot sizing

- Large number of identical jobs
- Continuous manufacturing in months or even years
- Long runs to make-to-stock, implying inventory holding costs
- *Setup time* and *setup costs* are significant
- Setup may be sequence dependent
- **Terminology**
  - jobs = items
  - sequence of identical jobs = run

Economic lot sizing

- **Objective**: **minimize total cost**
  - setup costs
  - inventory holding costs
- Optimal schedule
  - Trade-off between the two objectives
  - Cyclic schedules are used often
- **Applications**
  - Continuous manufacturing: chemical, paper, pharmaceutical.
  - Service industry: retail procurement (for each item)

Scheduling problem

- Determine the length of the runs (lot sizes)
  - gives lot sizes
- Determine the order of the runs
  - sequence to minimize setup cost
- **Economic Lot Scheduling Problem** (ELSP)

Overview

- **One type of item / one machine**
  - with and without setup time
- **Several types of items / one machine**
  - rotation schedules
  - arbitrary schedules
  - with / without sequence dependent setup times / cost
- **Generalizations to multiple machines**

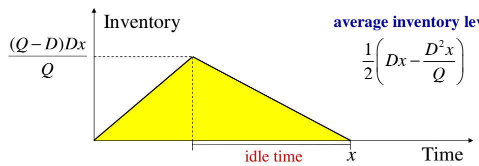
One type of item

- Single machine
- Single item type
- Production rate Q/time
- Demand rate D/time

➢ **Problem: determine the run length**

Minimize cost

- Let x denote the cycle time
- Demand over a cycle = Dx
- Length of production run needed = $Dx/Q = \rho\, x$
- Maximum inventory level = $(Q - D)Dx/Q$



Costs

- Setup cost is c and inventory holding cost per item per unit time is h.
- Average setup cost is c/x
- Average inventory holding cost:

$$\frac{1}{2}h\left(Dx - \frac{D^2 x}{Q}\right)$$

- Total cost

$$\frac{1}{2}h\left(Dx - \frac{D^2 x}{Q}\right) + \frac{c}{x}$$

Optimizing cost

- Solve $\min\left[\frac{1}{2}h\left(Dx - \frac{D^2 x}{Q}\right) + \frac{c}{x}\right]$
- Derivative with respect to x:

$$\frac{d}{dx}\left\{\frac{1}{2}h\left(Dx - \frac{D^2 x}{Q}\right) + \frac{c}{x}\right\} = \frac{1}{2}hD\left(1 - \frac{D}{Q}\right) - \frac{c}{x^2}$$

- Solving the minimization problem:

$$\frac{1}{2}hD\left(1 - \frac{D}{Q}\right) - \frac{c}{x^2} = 0$$

Optimal cycle time

$$\frac{1}{2}hD\left(1 - \frac{D}{Q}\right) = \frac{c}{x^2}$$

$$x^2 = \frac{2Qc}{hD(Q-D)}$$

$$\boxed{x = \sqrt{\frac{2Qc}{hD(Q-D)}}}$$

Optimal lot size

- Total production $Dx = \sqrt{\frac{2DQc}{h(Q-D)}}$
- When production capabilities are unlimited:

$$\sqrt{\frac{2DQc}{h(Q-D)}} \xrightarrow{Q\to\infty} \sqrt{\frac{2c}{hD}}$$

- Economic Lot Size (ELS) or Economic Order Quantity (EOQ):

$$\boxed{Dx = \sqrt{\frac{2Dc}{h}}}$$

Setup time

- Setup time s
- Idle time of a machine during a cycle: $x(1 - D/Q)$

$$\boxed{\rho = \frac{D}{Q} = \text{utilization of machine}}$$

- If $s \le x(1-\rho)$ solution is still optimal
- Otherwise cycle length $x = \dfrac{s}{1-\rho}$ is optimal, i.e. machine is never idle.

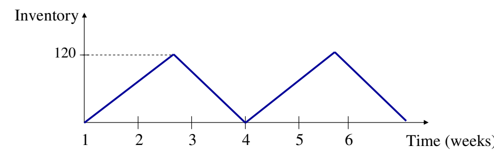
Example 7.2.1

- Production Q = 90/week
- Demand D = 50/week
- Setup cost c = 2000€
- Holding cost h = 20 €/item

$$x = \sqrt{\frac{2 \times 90 \times 2000}{20 \times 50 \times (90 - 50)}} = \sqrt{\frac{3600}{10 \times 40}} = \sqrt{\frac{36}{4}} = 3$$

Optimal schedule

- Cycle time = 3 weeks
- Lot size = Dx = 150 items
- Idle time = 3(1–5/9) = 1.33 weeks



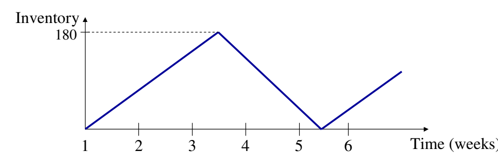
Example with setup times

- Now assume setup time
- If s < 1.33 weeks (about 9 days) then 3 weeks cycle is still optimal
- Otherwise the cycle time must be:

$$x = \frac{s}{1 - \rho}$$

- If setup last 2 weeks (maintenance and cleaning):
- x = 2/(1 – 5/9) = 4.5 weeks

Example with setup times



Example 7.2.2

- Q = 0.3333, D = 0.10, c = 90€, h = 5€
  - determine x, lot size

$$x = \sqrt{\frac{60}{0.5(0.3333 - 0.1)}} = 22.678$$

- Lot size: Dx = 2.2678.

- **What happens in a discrete setting?**

Example 7.2.2 (discrete)

- Time to produce one item is $p = 1/Q$ = 3 days.
- Demand rate is 1 item every 10 days.
- Lot size of k has to be produced every 10k days.
  - Total cost per day of lot size of 1 every 10 days is 90/10 = 9.
  - Total cost of lot size of 2 every 20 days is:

    (90 + 7×5)/20 = 6.25
  - Total cost of lot size of 3 every 30 days is:

    (90 + 7×5 + 14×5)/30 = 6.5
- So the optimal is to produce every 20 days a lot of 2.

Multiple items and rotation schedules

- Now assume n different items
- Demand rate for item j is D_j
- Production rate of item j is Q_j
- Setup independent of the sequence
- Length of production run of item j is $D_j x/Q_j$

- **Rotation schedule**: single run of each item

Scheduling decision

- Cycle length determines the run length for each item
- Only need to determine the cycle length x
- *Average inventory level* of item j:

$$\frac{1}{2}\left(D_j x - \frac{D_j^2 x}{Q_j}\right)$$

- With cost c_j, the *total average cost* per unit time is

$$\sum_{j=1}^{n}\left(\frac{1}{2}h_j\left(D_j x - \frac{D_j^2 x}{Q_j}\right) + \frac{c_j}{x}\right)$$

Optimal lot size

- Solving as in the previous case:

$$x = \sqrt{\left(\sum_{j=1}^{n}\frac{h_j D_j (Q_j - D_j)}{2Q_j}\right)^{-1}\sum_{j=1}^{n}c_j}$$

- Machine idle time during a cycle:

$$x\left(1 - \sum_{j=1}^{n}\frac{D_j}{Q_j}\right)$$

Optimal lot size

- Utilization factor of the machine due to item j:

$$\rho_j = \frac{D_j}{Q_j}$$

- With production capabilities unlimited ($Q_j \to \infty$):

$$x = \sqrt{\left(\sum_{j=1}^{n}\frac{h_j D_j}{2}\right)^{-1}\sum_{j=1}^{n}c_j}$$

Example 7.3.1

- Production rates, demand rates, holding costs and setup costs

items	1	2	3	4
D_j	50	50	60	60
Q_j	400	400	500	400
h_j	20	20	30	70
c_j	2000	2500	800	0

Optimal cycle length

$$x = \sqrt{\left(\sum_{j=1}^{n}\frac{h_j D_j (Q_j - D_j)}{2Q_j}\right)^{-1}\sum_{j=1}^{n}c_j}$$

$$= \sqrt{\left(2\times\frac{10\times 350}{8} + \frac{18\times 440}{10} + \frac{42\times 340}{8}\right)^{-1}5300}$$

$$= \sqrt{\left(\frac{10\times 350}{4} + \frac{18\times 440}{10} + \frac{42\times 340}{8}\right)^{-1}5300}$$

$$= \sqrt{(3452)^{-1}5300} = \sqrt{1.5353} = 1.24 \text{ months}$$

Solution

- Idle time is $0.48x = 0.595$ months.
- The total average cost per time unit is:

$$\sum_{j=1}^{n}\left(\frac{1}{2}h_j\left(D_j x - \frac{D_j^2 x}{Q_j}\right) + \frac{c_j}{x}\right) =$$

$$= 2155 + 2559 + 1627 + 2213 = 8554$$

- As the setup cost of item 4 is zero, a rotation schedule makes no sense here. We will address this problem later.

With setup times

- With sequence independent setup costs and no setup times the sequence within each lot does not matter

$\Rightarrow$ Only a lot sizing problem

- Even with setup times, if they are not job dependent then still only lot sizing

Job independent setup times

- If sum of setup times < idle time then our optimal cycle length remains optimal
- Otherwise we take it as small as possible.
- By increasing x until setup time is equal to idle time:

$$\sum_{j=1}^{n} s_j = x\left(1 - \sum_{j=1}^{n}\rho_j\right)$$

- The optimal x^* is given by:

$$x^* = \left(\sum_{j=1}^{n} s_j\right) \Bigg/ \left(1 - \sum_{j=1}^{n}\rho_j\right)$$

Job dependent setup times

- Now there is a sequencing problem
- **Objective**: minimize sum of setup times

➢ Equivalent to the Traveling Salesman Problem (TSP):

- A salesman must visit n cities exactly once with the objective of minimizing the total travel time, starting and ending in the same city.

Equivalence to TSP

- Item = city
- Travel time = setup time
- TSP is NP-hard
- If best sequence has

sum of setup times < idle time

$\Rightarrow$ optimal lot size and sequence

Long setup

- If **sum of setups > idle time**, then the optimal schedule has the property:
  - Each machine is either producing or being setup for production
- The sequence is obtained by applying the **Shortest Setup Time first (SST)**.
- *This is an extremely difficult problem with arbitrary setup times.*

Example 7.3.2

- Same data as Example 7.3.1 with setup times.
- Possible sequences as 3! = 6.
- **Setup times**:

items	1	2	3	4
s_{1k}	–	0.064	0.405	0.075
s_{2k}	0.448	–	0.319	0.529
s_{3k}	0.043	0.234	–	0.107
s_{4k}	0.145	0.148	0.255	–

Solution

- Recall that without setup times:
  - Cycle time is 1.24 months
  - Total idle time is 0.595 months
- Six sequences can be **enumerated** to find the best one.
- Possible solutions:
  - Sequence 1, 4, 2, 3 requires a total setup of 0.585 months and is optimal.
  - If SST is applied, sequence 1, 2, 3, 4 is selected. It requires a total setup of 0.635 months: *not optimal and exceeds idle time*!

Arbitrary schedules

- Sometimes a rotation schedule does not make sense (remember problem with no setup cost)
- For example, we might want to allow a cycle 1,4,2,4,3,4 if item 4 has no setup cost

➢ The problem is NP-hard: no efficient algorithms exist.

Problem formulation

- Assume sequence-independent setup cost and times
- Formulate as a nonlinear program:

$$\min_{\forall \text{ sequences}} \min_{\forall \text{ lot sizes}} \text{COST}$$

s.t.

demand met over the cycle

demand is met between production runs

Notation

- Recall that $\rho_j = D_j/Q_j$. A feasible solution exists iff:

$$\rho = \sum_{j=1}^{n} \rho_j < 1$$

- Setup cost and setup times

$$c_{jk} = c_k, \; s_{jk} = s_k.$$

- Define a sequence as:

$$j_1, \ldots, j_l, \ldots, j_\upsilon \; (\upsilon \geq n)$$

  - if $j_l = k$, then item k is produced in the l-th position of the sequence

Notation

- Item k produces in l-th position:

$$Q^l = Q_{j_l} = Q_k$$

- Setup cost: c^l,
- Setup time: s^l,
- Production time: τ^l
- Idle time: u^l

Inventory cost

- Let x be the cycle time
- Let v be the time between production of item k in j^l-th position:

$$v = \frac{Q^l \tau^l}{D^l} = \frac{Q_k \tau^l}{D_k}$$

- The highest inventory level is $(Q^l - D^l)\tau^l$, and the total inventory cost for item k is

$$\frac{1}{2} h^l (Q^l - D^l)\left(\frac{Q^l}{D^l}\right)(\tau^l)^2$$

Mathematical Program

$$\min_S \min_{x,\tau^l,u^l} \frac{1}{x}\left(\sum_{l=1}^{\upsilon} \frac{1}{2} h^l (Q^l - D^l)\left(\frac{Q^l}{D^l}\right)(\tau^l)^2 + \sum_{l=1}^{\upsilon} c^l\right)$$

subject to

$$\sum_{j \in I_k} Q_k \tau^j = D_k x, \qquad k = 1,\dots,n$$

$$\sum_{j \in L_l} (\tau^j + s^j + u^j) = \left(\frac{Q^l}{D^l}\right)\tau^l, \qquad l = 1,\dots,\upsilon$$

$$\sum_{j=1}^{\upsilon} (\tau^j + s^j + u^j) = x$$

Mathematical Program

- I_k is the set of all positions in the sequence in which item k is produced.
- L_l are all the positions in the sequence starting with position l (when item k is produced) up to the position in the sequence where item k is produced next.
- S is the set of all possible cyclic schedules.
- **1st constraint**: meet demand of item k over cycle
- **2nd constraint**: meet demand of item k over υ
- **3rd constraint**: cycle length

Two problems

- ELSP master problem
  - finds the best sequence $j_1, \dots, j_\upsilon$
- ELSP subproblem
  - finds the best production times, idle times, and cycle length (τ^l, u^l, x) given the sequence

➢ **Key idea: think of them separately!**

Subproblem (lot sizing)

- Sequence is fixed.

$$\min_{x,\tau^l,u^l} \frac{1}{x}\left(\sum_{l=1}^{v} \frac{1}{2} h^l (Q^l - D^l)\left(\frac{Q^l}{D^l}\right)(\tau^l)^2 + \sum_{l=1}^{v} c^l\right)$$

subject to

$$\sum_{j \in L_l} (\tau^j + s^j + u^j) = \left(\frac{Q^l}{D^l}\right)\tau^l, \qquad l = 1,\dots,v$$

$$\sum_{j=1}^{v} (\tau^j + s^j + u^j) = x$$

➢ But first: determine a sequence

Master problem

- Sequencing complicated
- Heuristic approach
- **Frequency Fixing and Sequencing** (**FFS**)
- Focus on how often to produce each item
- Phases:
  1. Computing relative frequencies
  2. Adjusting relative frequencies
  3. Sequencing

1. Computing relative frequencies

- Let y_k denote the number of times item k is produced in a cycle
- If runs of k are of equal length and equally spaced, frequency y_k and cycle time x determines run time τ_k:

$$\tau_k = \frac{\rho_k x}{y_k}$$

- We will
  - simplify the objective function by substituting

$$a_k = \frac{1}{2} h_k (Q_k - D_k) \rho_k$$

  - drop the second constraint $\Rightarrow$ sequence no longer important

Rewriting objective function

$$\min_{x,\tau^l,u^l} \frac{1}{x}\left(\sum_{l=1}^{\nu} \frac{1}{2} h^l (Q^l - D^l)\left(\frac{Q^l}{D^l}\right)(\tau^l)^2 + \sum_{l=1}^{\nu} c^l\right)$$

item k: y_k times
$$= \frac{1}{x}\left(\sum_{k=1}^{n} \frac{1}{2} y_k h_k (Q_k - D_k)\left(\frac{Q_k}{D_k}\right)\tau_k^2 + \sum_{k=1}^{n} c_k y_k\right)$$

substitute: a_k, ρ_k
$$= \frac{1}{x}\left(\sum_{k=1}^{n} a_k y_k \frac{\tau_k^2}{\rho_k^2} + \sum_{k=1}^{n} c_k y_k\right)$$

substitute: τ_k
$$= \sum_{k=1}^{n} \frac{a_k x}{y_k} + \sum_{k=1}^{n} \frac{c_k y_k}{x}$$

Assumption for each item

production runs of equal length and evenly spaced

Mathematical program

- Reduces to the nonlinear programming problem:

$$\min_{y_k, x} \sum_{k=1}^{n} \frac{a_k x}{y_k} + \sum_{k=1}^{n} \frac{c_k y_k}{x}$$

subject to
$$\sum_{k=1}^{n} \frac{s_k y_k}{x} \le 1 - \rho$$

Solution

- Using Lagrangean multiplier λ:

$$y_k = x\sqrt{\frac{a_k}{c_k + \lambda s_k}}$$

- Adjust cycle length for frequencies
- If there are idle times then $\lambda = 0$
- With no idle times, λ must satisfy

$$\sum_{k=1}^{n}\left(s_k \sqrt{\frac{a_k}{c_k + \lambda s_k}}\right) = 1 - \rho, \quad \text{since} \quad \sum_{k=1}^{n} s_k y_k = (1-\rho)x$$

2. Adjusting the frequencies

- Adjust the frequencies such that they are
  - integer
  - powers of 2
  - e.g. such that smallest $y_k = 1$
- ➢ Cost is within 6% of optimal cost
- **New frequencies and run times**:

$$y_k' \text{ and } \tau_k'$$

3. Sequencing

- Variation of LPT
- Let $\quad y'_{\max} = \max\left(y'_1, \ldots, y'_n\right)$
- Consider the problem with $y'_{\max}$ machines in parallel and y'_k jobs of length τ'_k **evenly spaced**
- Example: when $y'_{\max} = 6$, and $y'_k = 3$, then there are two choices:
  - assign 3 jobs to machines (1,3,5) or to (2,4,6)

3. Sequencing

- List pairs $(y_k^{'}, \tau_k^{'})$ in decreasing order
- Pairs with equal $y_k^{'}$ are listed in decreasing order of processing time $\tau_k^{'}$
- Schedule one at a time considering spacing

- Equal lot sizes is possible only if for all machines:

$$\textit{assigned processing time} < \frac{x}{y^{'}_{\max}}$$

Example 7.4.1

- Consider example 7.3.1 *without rotation schedule*.

items	1	2	3	4
D_j	50	50	60	60
Q_j	400	400	500	400
h_j	20	20	30	70
c_j	2000	2500	800	0

Example (cont.)

- From Ex. 7.3.1: $(1-\rho)\,x = 0.48\,x$
- As: $$a_k = \frac{1}{2} h_k (Q_k - D_k) \rho_k$$
- We can compute:

items	1	2	3	4
D_j	50	50	60	60
Q_j	400	400	500	400
h_j	20	20	30	70
c_j	2000	2500	800	0
ρ_j	**0.125**	**0.125**	**0.12**	**0.15**
a_j	**437.5**	**437.5**	**792**	**1785**

Example (cont.)

- $y_k = \sqrt{\dfrac{a_k}{c_k + \lambda s_k}}\, x$, we consider no idle times ($\lambda = 0$).
- It follows that:
  - $y_1 = 0.47\,x$
  - $y_2 = 0.42\,x$
  - $y_3 = 0.99\,x$
  - $y_4 = \infty$
- Suppose that cycle time x is 2 months.
- Approximate values: $y_1 = y_2 = 1$, $y_3 = 2$, $y_4 = 16$.

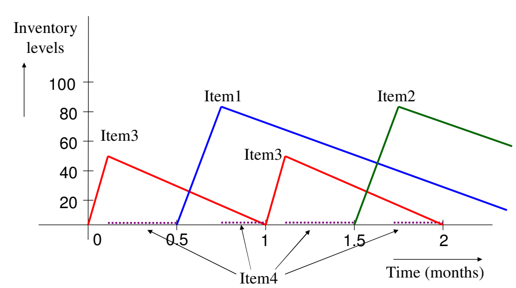
Example (cont.)

- Run times are $\tau_k = \dfrac{\rho_k x}{y_k}$:

- $\tau_1 = \tau_2 = 0.25$, $\tau_3 = 0.12$, $\tau_4 = 0.3/16$.
- Application of LPT heuristic
  - Number of machines in parallel $y_{\max} = 16$.
  - Item 4 is assigned to all 16 machines with processing times 0.3/16.
  - Item 3 is assigned to machines 1 and 9.
  - Item 2 is put in machine 5 and item 1 in machine 13.
  - Cyclic schedule: |4,3|4|4|4|4,1|4|4|4|4,3|4|4|4|4,2|4|4|4|4|

Example: another solution

- Minimum level of y_4 is 4. Sequence is: 4,3,4,1,4,3,4,2.



Example: another solution

- The total average cost per time unit is:
  $1875 + 2125 + 1592 + 190 = 5782$
- It was 8554 in Example 7.3.1!

Example 7.4.2: with setup times

- The setup times s_j are sequence independent.

items	1	2	3	4
D_j	50	50	60	60
Q_j	400	400	500	400
h_j	20	20	30	70
c_j	2000	2500	800	0
s_j	**0.5**	**0.2**	**0.1**	**0.2**
ρ_j	0.125	0.125	0.12	0.15
a_j	437.5	437.5	792	1785

Solution

- Item 4 cannot be arbitrarily high as before.
- Find λ that satisfies:

$$\sum_{k=1}^{n}\left(s_k\sqrt{\frac{a_k}{c_k+\lambda s_k}}\right)=1-\rho$$

- It has the value λ ≈ 8000.
- The frequencies are given by

$$y_k = x\sqrt{\frac{a_k}{c_k+\lambda s_k}}$$

Solutions

- Thus: $y_1 = 0.27x$, $y_2 = 0.33x$, $y_3 = 0.70x$ and $y_4 = 1.05x$.
- If cycle time is three months, solutions can be:
  (1,1,2,2) or (1,1,2,4), all power of 2
- Solution (1,1,2,2) has the sequence 1,3,4,2,3,4
  - Idle time before considering setups is 0.48×3 = 1.44
  - Total amount of setup time required is 1.3
  - ➢ Schedule is feasible.
  - ➢ Cycle time can be slightly reduced.

Solutions (concl.)

- Solution (1,1,2,4) has the sequence 1,4,3,4,2,4,3,4
  - Total amount of setup time required is 1.7 and **schedule is not feasible** in a cycle length of 3 months.
  - The cycle length should be larger (see Exercise 7.10)

More general ELSP models

- So far, all models are single machine models
- Extensions to multiple machines
  - parallel machines
  - flow shop
  - flexible flow shop

Parallel machines

- m identical machines in parallel
- There are setup cost but no setup time
- Item process on only one of the m machines
- For item j, utilization factor is again $\rho_j = D_j/Q_j$.
- Condition for a feasible solution is:

$$\sum_{j=1}^{n} \rho_j \leq m$$

Decision variables

- Assume
  - rotation schedule
  - equal cycle for all machines
- Same as previous multi-item problem
- Addition: assignment of items to machines

- **Objective**: balance the load
- Use heuristic LPT with ρ_j as processing times.

Different cycle lengths

- Allow different cycle lengths for machines
- Intuition: should be able to reduce cost

- **Objective**: assign items to machines to balance the load
- Complication: should not assign items that favor short cycle to the same machine as items that favor long cycle

Heuristic balancing

- Compute cycle length for each item
- Rank in decreasing order
- Allocate jobs sequentially to the machines until capacity of each machine is reached
- Adjust balance if necessary

Further generalizations

- **Sequence dependent setup**
- Must consider:
  - preferred cycle time
  - machine balance
  - setup times
- Problem is unsolved
- General schedules $\Rightarrow$ **even harder**!
- This problem requires more research!

Flow shop

- Machines configured in series
- Assume no setup time
- Assume production rate of each item is identical for every machine

  $\Rightarrow$ Can be synchronized
- *Problem is reduced to single machine problem* with setup cost:

$$c_j = \sum_{i=1}^{m} c_{ij}$$

Variable production rates

- Production rate for each item *not* equal for every machine
- Difficult problem
- Little research

- **Flexible flow shop**: need even more stringent conditions

Discussion

- Lot sizing models
  - demand assumed known, which determines throughput
  - make-to-stock systems: due date of little importance/not available
  - Objective: minimize inventory and setup costs (time).
- Practical problems are a combination of make-to-stock and make-to-order.
  - In these problems facilities are set up in series. This area of research is known as:

Supply Chain Management